Celestial Mech Dyn Astr (2006) 94:399–410
DOI 10.1007/s10569-006-9001-5

ORIGINAL ARTICLE

Two-Way Orbits

Ossama Abdelkhalik · Daniele Mortari

Received: 9 March 2005 / Accepted: 30 January 2006 /
Published online: 27 April 2006

Abstract This paper introduces a new set of compatible orbits called "Two-Way Orbits," whose ground track path is a closed-loop trajectory that intersects at certain points with tangent intersections. The spacecraft passes over these tangent intersections once in a prograde mode and once in a retrograde mode. Motivations are found for the need to have simultaneous observations of the same target area in both Earth observation and reconnaissance systems. The general mathematical model to design a Two-Way Orbit is presented for the specific case where the tangent points are experienced at the orbit extremes, perigee and apogee. As for the general case, Two-Way Orbit conditions are formulated and numerically solved. Results show that, in general, Two-Way Orbits could be formed over any point on Earth. Since Two-Way Orbits use compatible orbits, the theory of Flower Constellations can be applied to them. Using these Two-Way Orbits, this paper also introduces the Two-Way Flower Constellations that have one spacecraft prograde and one retrograde passing simultaneously over the tangent intersection.

Keywords New orbits · Compatible orbits · Satellite constellations

1. Introduction

With the introduction of Flower Constellations (FC) (Mortari et al. 2004a) newly unified family of satellite constellations characterized by compatible orbits and axial-symmetric dynamics has been created. The most interesting feature of this novel methodology is that for some particular value of the design parameters, the entire constellation forms a rigid object (e.g., a triangle, square, circle, 3-D ellipsoid, star, or more complex shapes) that rotates

Ossama Abdelkhalik (✉) · Daniele Mortari
Department of Aerospace Engineering, Texas A&M University, H.R. Bright Building,
3141 TAMU, College Station, Texas 77843-3141, USA
Tel.: +1-979-458-0550
e-mail: omar@tamu.edu

Daniele Mortari
Tel.: +1-979-845-0734
e-mail: mortari@tamu.edu

about the constellation axis of symmetry with constant angular velocity. The characteristic dynamics of an FC preserve the shape of this novel space object (time invariant) and the orientation of this space object can be freely chosen.

Flower Constellations open a new frontier on complex satellite formations for two main reasons. First, Flower Constellations can be seen as constituted of two distinct parts: an "internal part," associated with the motion of all the satellites along a prescribed identical relative space track, and an "external part," associated with the dynamic of the whole constellation, as a rigid *object*, rigidly spinning with constant angular velocity. Second, these new constellation objects are used as *building blocks* to construct more complex configurations – with extremely promising solutions – to accomplish more complex tasks.

Flower Constellations, and the more recently introduced *Synodic* and *Relative Flower Constellations* (Mortari et al. 2005), combine a number of new attractive features suitable for many potential classical applications (communications, Earth and deep space observation, coverage, navigation systems, etc.), as well as for new and advanced concepts.

A Flower Constellation is built using orbits that are compatible with respect to an assigned rotating reference frame. A compatible orbit is an orbit which trajectory with respect to an assigned rotating frame is a closed trajectory. This compatibility implies that all the spacecraft in this rotating frame follow the same continuous closed-loop trajectory. In particular, when the reference frame is chosen to be Earth-Centered Earth-Fixed (ECEF), then the FC spacecraft all follow the same relative trajectory, space track, in the ECEF frame and, consequently, the same continuous closed-loop ground track. For information on Flower Constellations see also Mortari et al. (2004b), Park et al. (2004), Park et al. (2005), Wilkins (2004a), Wilkins et al. (2004b) and Wilkins et al. (2004c).

In general, the ground track is made of prograde and retrograde segments where the spacecraft ground track longitude is increasing or decreasing with time, respectively. Also, the ground track (for a compatible orbit) is a continuous closed-loop line that intersects itself at several points. These intersections can be characterized by the angle between the ground velocities along the two intersecting parts (see Fig. 1). When this angle is equal to π, then the intersecting point is a tangent intersecting point and the two intersecting parts are one prograde and another retrograde over the intersecting point on the Earth's surface. This tangential relationship describes the concept of Two-Way Orbits.

Fig. 1 Ground track intersecting angle

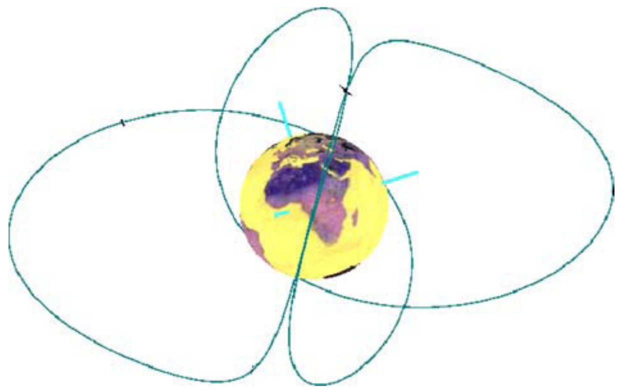

Fig. 2 Two-Way Orbits example

In Two-Way Orbits the relative trajectory will have at least one tangent intersecting point. This intersection implies that it is possible to build special Flower Constellations with one spacecraft moving along the tangent prograde direction and another spacecraft moving along the tangent retrograde direction. In particular, it is possible to phase the spacecraft in a way such that they will pass over the tangent point simultaneously (see Fig. 2).

Two cases will be considered. The first is the "special" Two-Way Orbit where the tangential intersection point is at the perigee of one spacecraft and the apogee of the other. The second case is the "general" Two-Way Orbits where the tangential intersection point is any general point on the trajectories of the two spacecraft.

In this paper, we derive conditions on the orbits' parameters such that it constitutes a Two-Way Orbit. The second section briefly reviews the theory of Flower Constellations. The third section reviews the orbit compatibility conditions. In the fourth section, the case of special Two-Way Orbits is considered. A Two-Way Orbit condition on the orbit inclination is derived. A plot is generated relating the inclination versus the eccentricity for specified values of the semimajor axis. The fifth section develops similar analysis for general Two-Way Orbits. However, for the general case the solution is obtained numerically. An algorithm for the numerical solution is presented in the sixth section. The last section considers the compatibility of the developed conditions with Flower Constellation theory. Results show that the derived conditions are compatible with the FCs only in the *special Two-Way Orbits*.

2. Compatible orbits

Consider an Earth-Centered Earth-Fixed (ECEF) system of coordinates identified by $\mathcal{E} = \{\mathcal{O},\ \hat{e}_x,\ \hat{e}_y,\ \hat{e}_z\}$, where the origin $\mathcal{O}$ is at the center of the Earth, $\hat{e}_x$ on the equatorial plane at Greenwich meridian, $\hat{e}_z$ is aligned with Earth's spin axis, and $\hat{e}_y = \hat{e}_z \times \hat{e}_x$ to form a right-handed reference frame.

An orbit is called *compatible* with respect to the Earth (Carter 1991) when the spacecraft trajectory in $\mathcal{E}$ constitutes a closed-loop relative trajectory. A compatible orbit, which is often and inappropriately called *repeated ground track* orbit[1], is defined as the orbit whose orbital nodal period T_Ω (node to node) satisfies the relationship

[1] Any two equatorial orbits have the same repeated ground track but, in general, they do not follow the same relative trajectory in ECEF; that is, they are not, in general, compatible.

$$N_p\, T_\Omega = N_d\, T_{\Omega G}, \tag{1}$$

where N_p and N_d are two integer numbers indicating the number of orbit periods and the number of the Earth rotational periods to repeat, and where $T_{\Omega G}$ is the Greenwich nodal period, which has been defined by Carter (1991) as

$$T_{\Omega G} = \frac{2\pi}{\dot{\alpha}_\oplus - \dot{\Omega}}, \tag{2}$$

where $\dot{\alpha}_\oplus = 7.29211585530 \times 10^{-5}$ rad/sec is the rotation rate of the Earth and $\dot{\Omega}$ is the nodal regression of a satellite's orbit plane caused by perturbations such as the Earth's oblateness. In particular, $T_r = N_p\, T_\Omega$ is the period of repetition on the relative trajectory.

(1) and (2) allow us to write

$$T_\Omega = \left(\frac{2\,\pi}{\dot{\alpha}_\oplus - \dot{\Omega}}\right) \frac{N_d}{N_p} = \left(\frac{2\,\pi}{\dot{\alpha}_\oplus - \dot{\Omega}}\right) \xi, \tag{3}$$

where $\xi = N_d/N_p$ is the rational *compatibility parameter*. (3) tells us that, for every distinct value of ξ, there is a different nodal period T_Ω, associated with Earth's compatible orbits. However, this equation can also be seen from a different perspective: for a given value of ξ, an arbitrary orbit (with nodal period T_Ω and nodal rate $\dot{\Omega}$) can be seen as *compatible* with a *fictitious Earth* that rotates with angular velocity

$$\dot{\alpha} = \dot{\Omega} + \frac{2\,\pi}{T_\Omega}\,\xi. \tag{4}$$

Therefore, every orbit can be seen as *compatible* with an associated Earth-Centered Rotating (ECR) system of coordinates that rotates at the angular velocity provided by (3). The final result is that any Earth-compatible orbit is compatible with an infinite The compatibility concept is a relative concept, which refers to a rotating reference frame. Thus, if we consider a different rotating reference frame, then there will be a definition of *orbit compatibility* with respect to *this* reference frame.

3. The "special" Two-Way Orbits

The condition for two satellites to have tangent ground tracks at a point is to have parallel Earth-relative velocities at that point. The Earth-relative velocity, $\vec{v}$, is the velocity of the satellite with respect to an Earth rotating system of coordinates.

$$\vec{v} = \vec{V} - \vec{V}_E, \tag{5}$$

where $\vec{V}$ is the satellite velocity in Earth-Centered Inertial (ECI) reference frame and $\vec{V}_E$ is the local geographical velocity evaluated at radius $\vec{r}$ in ECI. The transformation matrix between inertial and orbital reference frames is ($C_\Omega \equiv \cos\Omega$, $S_\Omega \equiv \sin\Omega$, and so on)

$$R^{\mathrm{T}} = \begin{bmatrix} C_\Omega C_\omega - C_i S_\Omega S_\omega & -C_\Omega S_\omega - C_i S_\Omega C_\omega & S_i S_\Omega \\ S_\Omega C_\omega + C_i C_\Omega S_\omega & -S_\Omega S_\omega + C_i C_\Omega C_\omega & -S_i C_\Omega \\ S_\omega S_i & C_\omega S_i & C_i \end{bmatrix}. \tag{6}$$

This matrix rotates vectors from the perifocal frame (r_o) to the inertial frame (r_i), and vice versa

$$r_i = R^{\mathrm{T}}\, r_o \quad \Longleftrightarrow \quad r_o = R\, r_i. \tag{7}$$

In particular, position and velocity are transformed accordingly with

$$r_i = \frac{p}{1 + e\,\cos\varphi} \left\{ \begin{array}{c} \cos\Omega\,\cos(\omega+\varphi) - \sin\Omega\,\sin(\omega+\varphi)\,\cos i \\ \sin\Omega\,\cos(\omega+\varphi) + \cos\Omega\,\sin(\omega+\varphi)\,\cos i \\ \sin(\omega+\varphi)\,\sin i \end{array} \right\}, \tag{8}$$

while the velocity in the orbital reference frame is expressed as

$$v_o = \sqrt{\frac{\mu}{p}} \left\{ \begin{array}{c} -\sin\varphi \\ e + \cos\varphi \\ 0 \end{array} \right\}. \tag{9}$$

Let us consider, for simplicity, two eccentric orbits having the apsidal lines lying on the equatorial plane ($\omega = 0$). Under this condition, the velocity in ECI of the first orbit at perigee is

$$\vec{V}_{p1} = (e+1)\sqrt{\frac{\mu}{p}} \left\{ \begin{array}{c} -\sin\Omega_1\,\cos i \\ \cos\Omega_1\,\cos i \\ \sin i \end{array} \right\}, \tag{10}$$

while at apogee of the second orbit the velocity in ECI is

$$\vec{V}_{a2} = (e-1)\sqrt{\frac{\mu}{p}} \left\{ \begin{array}{c} -\sin\Omega_2\,\cos i \\ \cos\Omega_2\,\cos i \\ \sin i \end{array} \right\}. \tag{11}$$

The local geographical velocity is

$$\vec{V}_E = \vec{\omega}_E \times \vec{r}, \tag{12}$$

where $\vec{\omega}_E$ is the Earth angular velocity. Specializing (12) for orbit #1, we obtain the expression for the Earth-relative velocity at perigee

$$\vec{V}_{Ep1} = \left\{ \begin{array}{c} 0 \\ 0 \\ \omega_E \end{array} \right\} \times \frac{p}{1+e} \left\{ \begin{array}{c} \cos\Omega_1 \\ \sin\Omega_1 \\ 0 \end{array} \right\} = \frac{p\,\omega_E}{e+1} \left\{ \begin{array}{c} -\sin\Omega_1 \\ \cos\Omega_1 \\ 0 \end{array} \right\}, \tag{13}$$

while the Earth-relative velocity at apogee of orbit #2 is

$$\vec{V}_{Ea2} = \left\{ \begin{array}{c} 0 \\ 0 \\ \omega_E \end{array} \right\} \times \frac{p}{1-e} \left\{ \begin{array}{c} -\cos\Omega_2 \\ -\sin\Omega_2 \\ 0 \end{array} \right\} = \frac{p\,\omega_E}{e-1} \left\{ \begin{array}{c} -\sin\Omega_2 \\ \cos\Omega_2 \\ 0 \end{array} \right\}. \tag{14}$$

Substituting (10) and (13) into (5) we obtain

$$\vec{v}_{p1} = (e+1)\sqrt{\frac{\mu}{p}} \left\{ \begin{array}{c} -\sin\Omega_1\,\cos i \\ \cos\Omega_1\,\cos i \\ \sin i \end{array} \right\} - \frac{p\,\omega_E}{e+1} \left\{ \begin{array}{c} -\sin\Omega_1 \\ \cos\Omega_1 \\ 0 \end{array} \right\} \tag{15}$$

and substituting (11) and (14) into (5) we obtain

$$\vec{v}_{a2} = (e-1)\sqrt{\frac{\mu}{p}} \left\{ \begin{array}{c} -\sin\Omega_2\,\cos i \\ \cos\Omega_2\,\cos i \\ \sin i \end{array} \right\} - \frac{p\,\omega_E}{e-1} \left\{ \begin{array}{c} -\sin\Omega_2 \\ \cos\Omega_2 \\ 0 \end{array} \right\}. \tag{16}$$

In general, in order to have tangent ground tracks at the intersection, the two velocity vectors and the vector pointing at the intersection, $\vec{R}_{eq}$, must be linearly dependent (they identify the plane passing through the origin of the coordinates and containing the two velocities $\vec{v}_{p1}$ and $\vec{v}_{a2}$). In our case, since we are looking for tangency at the equator and at the perigee of one orbit and the apogee of the other orbit, then our tangential condition can be substituted with the condition for the two velocity vectors, $\vec{v}_{p1}$ and $\vec{v}_{a2}$), being parallel. This parallelism implies that we can write

$$\vec{v}_{a2} = k\,\vec{v}_{p1}, \tag{17}$$

where k is the proportionality constant. Let $v_{a2}(i)$ be the ith component of $\vec{v}_{a2}$ and $v_{p1}(i)$ be the ith component of $\vec{v}_{p1}$, then from (17):

$$\frac{v_{p1}(1)}{-v_{a2}(1)} = \frac{v_{p1}(2)}{-v_{a2}(2)}, \tag{18}$$

$$\frac{v_{p1}(1)}{-v_{a2}(1)} = \frac{v_{p1}(3)}{-v_{a2}(3)}, \tag{19}$$

and

$$k = \frac{v_{a2}(3)}{v_{p1}(3)} = \frac{e-1}{e+1}. \tag{20}$$

Substituting from (15) and (16) into (18) and (19), we obtain

$$\frac{-C_i\,S_{\Omega 1}\,V_p + \omega_E\,r_p\,S_{\Omega 1}}{C_i\,S_{\Omega 2}\,V_a - \omega_E\,r_a\,S_{\Omega 2}} = \frac{C_i\,C_{\Omega 1}\,V_p - \omega_E\,r_p\,C_{\Omega 1}}{-C_i\,C_{\Omega 2}\,V_a + \omega_E\,r_a\,C_{\Omega 2}} \tag{21}$$

and

$$\frac{-C_i\,S_{\Omega 1}\,V_p + \omega_E\,r_p\,S_{\Omega 1}}{C_i\,S_{\Omega 2}\,V_a - \omega_E\,r_a\,S_{\Omega 2}} = -\frac{1}{k}, \tag{22}$$

respectively, where

$$V_p = (1+e)\sqrt{\frac{\mu}{p}} \quad \text{and} \quad V_a = (1-e)\sqrt{\frac{\mu}{p}} \tag{23}$$

represent the modulus of the velocity at perigee and apogee, respectively. The semi-latus rectum, p, can be expressed as a function of the perigee altitude, apogee altitude, or the semi-major axis as (Sidi 1997):

$$p = r_p(1+e) = r_a(1-e) = a(1-e^2). \tag{24}$$

Substituting from (24) into (23) we obtain

$$V_p = \sqrt{\frac{2\mu}{r_p} - \frac{\mu}{a}} \quad \text{and} \quad V_a = \sqrt{\frac{2\mu}{r_a} - \frac{\mu}{a}}. \tag{25}$$

With little manipulation, (21) is satisfied if

$$\sin(\Omega_1 - \Omega_2) = 0 \tag{26}$$

or

$$V_a\,V_p\,C_i^2 - \omega_E\,C_i\,(r_p\,V_a + r_a\,V_p) + \omega_E^2\,r_p\,r_a = 0. \tag{27}$$

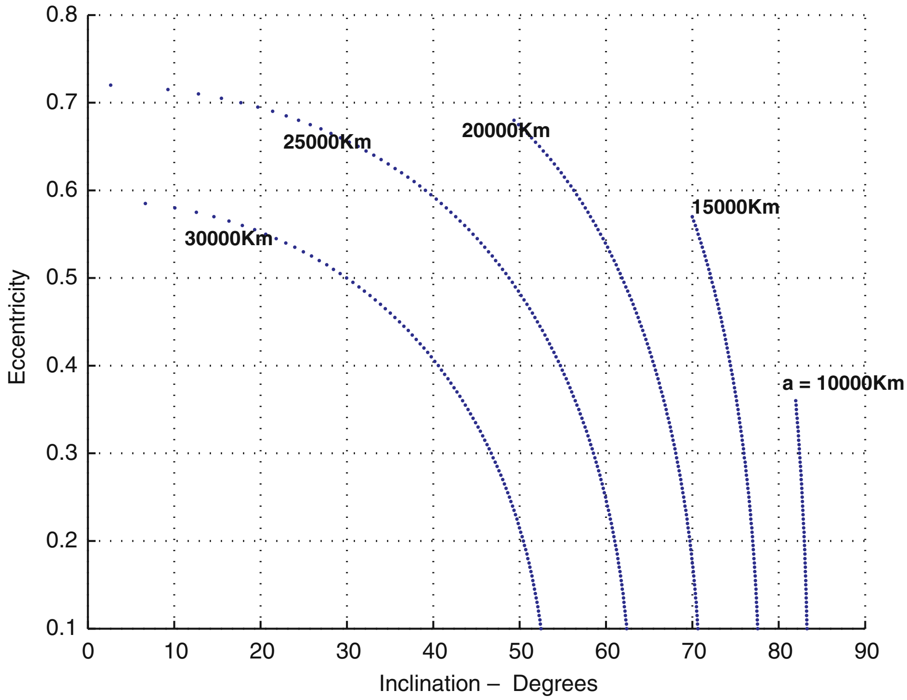

Fig. 3 Two-Way Orbits' inclinations for different values of e and a

The latter case is refused because it does not satisfy the condition in (22). The result in (26) states that either $\Omega_1 = \Omega_2$, which is a trivial case where the two orbits are identical, or

$$\Omega_1 - \Omega_2 = \pi. \tag{28}$$

The latter case gives the condition on the right ascension of the ascending node for the two orbits. The condition in (22), after manipulation, implies that

$$\cos i = \frac{\omega_E \left(\dfrac{r_p}{V_p} S_{\Omega 1} - \dfrac{r_a}{V_a} S_{\Omega 2} \right)}{S_{\Omega 1} - S_{\Omega 2}}. \tag{29}$$

From (28), we have $\sin(\Omega_2) = -\sin(\Omega_1)$ then (29) simplifies to

$$\cos i = \frac{\omega_E}{2} \left(\frac{r_p}{V_p} + \frac{r_a}{V_a} \right) \tag{30}$$

(28) and (30) constitute the necessary and sufficient conditions to have Two-Way Orbits.

Figure 3 shows *Two-Way* Orbits' inclinations for different values of e and a. As the semi-major axis decreases, the inclination angle increases, and vice versa, for a given eccentricity of the orbits. For a given semi-major axis, the lower the eccentricity of the orbits, the higher is the inclination.

4. The "general" Two-Way Orbits

We considered the case where the two orbits have similar shape, size and inclination. Moreover, we assumed zero argument of perigee and the intersection point occurs at the perigee and apogee points of the two orbits. In this section we look at the same problem but the intersection point is any general point, not necessarily an apogee or perigee.

First, we find the condition of having an intersection between the two ground tracks for two different orbits. An intersection between the ground tracks occurs if the two position vectors of the two satellites are parallel.

$$\vec{r}_1 = k_r \vec{r}_2. \tag{31}$$

Substituting for the vectors $\vec{r}_1$ and $\vec{r}_2$ from (8) then we can get the following two conditions for an intersection to occur

$$\frac{\cos\Omega_1 \cos(\omega+\varphi_1) - \sin\Omega_1 \sin(\omega+\varphi_1)\cos i}{\cos\Omega_2 \cos(\omega+\varphi_2) - \sin\Omega_2 \sin(\omega+\varphi_2)\cos i} = \frac{\sin(\omega+\varphi_1)}{\sin(\omega+\varphi_2)} \tag{32}$$

and

$$\frac{\sin\Omega_1 \cos(\omega+\varphi_1) + \cos\Omega_1 \sin(\omega+\varphi_1)\cos i}{\sin\Omega_2 \cos(\omega+\varphi_2) + \cos\Omega_2 \sin(\omega+\varphi_2)\cos i} = \frac{\sin(\omega+\varphi_1)}{\sin(\omega+\varphi_2)}. \tag{33}$$

These two conditions can be simplified to the following form

$$\begin{bmatrix} C_i a_1 & -a_3 \\ a_3 & C_i a_1 \end{bmatrix} \begin{Bmatrix} S_{\Omega 1} \\ C_{\Omega 1} \end{Bmatrix} = \begin{bmatrix} C_i a_1 & -a_4 \\ a_4 & C_i a_1 \end{bmatrix} \begin{Bmatrix} S_{\Omega 2} \\ C_{\Omega 2} \end{Bmatrix}, \tag{34}$$

where, $a_1 = \sin(\omega+\varphi_1)\sin(\omega+\varphi_2)$, $a_3 = \cos(\omega+\varphi_1)\sin(\omega+\varphi_2)$, and $a_4 = \sin(\omega+\varphi_1)\cos(\omega+\varphi_2)$.

4.1. Observations

1. We notice that for the special case where $a_3 = a_4$, then either $\Omega_1 = \Omega_2$, which is the obvious case, or ${C_i}^2 {a_1}^2 + {a_3}^2 = 0$ which is satisfied only if $\omega + \varphi_2 = n\pi$ where, $n = 0, 1, \ldots$. This latter case is the special case solved in the previous section.
2. If we eliminate C_i from (34) then we get

$$\frac{a_4 S_{\Omega 2} - a_3 S_{\Omega 1}}{\Delta C} = \frac{a_3 C_{\Omega 1} - a_4 C_{\Omega 2}}{\Delta S}, \tag{35}$$

   where $\Delta C = C_{\Omega 1} - C_{\Omega 2}$ and $\Delta S = S_{\Omega 1} - S_{\Omega 2}$. By rearrangement of (35) we can write

$$\frac{a_4}{a_3} \equiv \frac{\tan(\omega+\varphi_1)}{\tan(\omega+\varphi_2)} = \frac{C_{\Omega 1}\Delta C + S_{\Omega 1}\Delta S}{C_{\Omega 2}\Delta C + S_{\Omega 2}\Delta S}. \tag{36}$$

   This expression can be further simplified to the form

$$(a_3 + a_4)[\cos(\Omega_1 - \Omega_2) - 1] = 0, \tag{37}$$

   which means that either $\Omega_1 = \Omega_2$, which is a trivial solution, or $a_3 = a_4$. The latter can be written in the form

$$\omega_1 + \varphi_1 + \omega_2 + \varphi_2 = n\pi \qquad n = 0, 1, \ldots \tag{38}$$

3. For the special case where $\Omega_1 - \Omega_2 = \pi$, the above equation reduces to $a_4/a_3 = -1$. This relationship means that $\varphi_2 = \pi - \varphi_1$, which is the special case solved in the previous section.

The condition in (38) implies that the two trajectories of two satellites intersect at a certain point; however, it does not imply that both of the two satellites will pass by this point at the same time. To guarantee that both will pass by the intersection point at the same time, we introduce the following condition.

Assume the two satellites of interest intersect at time $t = t_i$, and writing the time equation for both satellites at the intersection point

$$\left(t_i - t_{p1}\right) n = \psi_{1i} - e \sin(\psi_{1i}), \tag{39}$$

$$\left(t_i - t_{p2}\right) n = \psi_{2i} - e \sin(\psi_{2i}), \tag{40}$$

where n is the mean motion. Then,

$$\psi_{2i} - e \sin(\psi_{2i}) = \psi_{1i} - e \sin(\psi_{1i}) - n(t_{p2} - t_{p1}). \tag{41}$$

In order to find t_{p1} and t_{p2}, we need to define the phasing between the two satellites. Recalling that for a *Flower Constellation* we have two phasing conditions(Mortari et al. 2004a). The first, for a two-body problem case, is

$$M_{k+1}(0) = M_k(0) + 2\pi \frac{F_n}{F_d} \frac{n}{\omega_E}, \tag{42}$$

where F_n and F_d are integers defining the phasing of the two satellites. Now if we set, without loss of generality, $t_{p1} = -M_1(0)/n$ and $t_{p2} = -M_2(0)/n$, then the condition in (41) becomes

$$\psi_{2i} - e \sin(\psi_{2i}) = \psi_{1i} - e \sin(\psi_{1i}) + 2\pi \frac{F_n}{F_d} \frac{n}{\omega_E}. \tag{43}$$

(38) and (43) completely determine the intersection point of the two satellites.

Now, we proceed to the condition of Two-Way orbits. We proceed as in the previous section but with general orbit parameters. Assume that the point of intersection occurs at point *1* in the first orbit corresponding to a true anomaly, φ_1, and at point *2* in the second orbit corresponding to a true anomaly, φ_2. Assume also that the two orbits have common e, i, and ω. Then it can be shown that the velocity of point *1* relative to the Earth is

$$\vec{V}_1^R = \begin{Bmatrix} \sqrt{\dfrac{\mu}{p_1}} \left(\tau_{11}\tau_{12} + \tau_{13}\tau_{14}\right) + \dfrac{\omega_E p_1}{1 + e\cos(\varphi_1)} \tau_{15} \\ \sqrt{\dfrac{\mu}{p_1}} \left(\tau_{21}\tau_{12} + \tau_{23}\tau_{14}\right) - \dfrac{\omega_E p_1}{1 + e\cos(\varphi_1)} \tau_{25} \\ \sqrt{\dfrac{\mu}{p_1}} S_i \left(S_\omega \tau_{12} + C_\omega \tau_{14}\right) \end{Bmatrix}, \tag{44}$$

where

$$\begin{aligned}
\tau_{11} &= C_{\Omega 1} C_\omega - C_i S_{\Omega 1} S_\omega \\
\tau_{12} &= \sin(\varphi_1) \left[\cos(\varphi_1)(1 - e) - 1\right] \\
\tau_{13} &= -C_{\Omega 1} S_\omega - C_i S_{\Omega 1} C_\omega \\
\tau_{14} &= 1 - \left[\cos(\varphi_1)(1 - e) - 1\right] \cos(\varphi_1) \\
\tau_{15} &= S_{\Omega 1} \cos(\omega + \varphi_1) + C_i C_{\Omega 1} \sin(\omega + \varphi_1) \\
\tau_{21} &= S_{\Omega 1} C_\omega + C_i C_{\Omega 1} S_\omega \\
\tau_{23} &= -S_{\Omega 1} S_\omega + C_i C_{\Omega 1} C_\omega \\
\tau_{25} &= C_{\Omega 1} \cos(\omega + \varphi_1) - C_i S_{\Omega 1} \sin(\omega + \varphi_1)
\end{aligned}$$

The vector $\vec{V}_2^R$ is defined similar to $\vec{V}_1^R$ with p_2, Ω_2 and φ_2 replacing p_1, Ω_1 and φ_1, respectively.

$$\vec{V}_2^R = \begin{Bmatrix} \sqrt{\frac{\mu}{p_2}}\,(\sigma_{11}\,\sigma_{12} + \sigma_{13}\sigma_{14}) + \frac{\omega_E\, p_2}{1 + e\cos(\varphi_2)}\sigma_{15} \\ \sqrt{\frac{\mu}{p_2}}\,(\sigma_{21}\sigma_{12} + \sigma_{23}\sigma_{14}) - \frac{\omega_E\, p_2}{1 + e\cos(\varphi_2)}\sigma_{25} \\ \sqrt{\frac{\mu}{p_2}} S_i\,(S_\omega\sigma_{12} + C_\omega\sigma_{14}) \end{Bmatrix}, \tag{45}$$

where σ_{ij} corresponds to τ_{ij}.

The condition for having Two-Way orbits is that the vectors $\vec{V}_1^R$, $\vec{V}_2^R$ and the position vector of the intersection point, $\vec{r}_i$, belong to the same plane. Then we can write this condition as follows:

$$\chi = \vec{r}_i \cdot \left(\vec{V}_1^R \times \vec{V}_2^R\right) = 0. \tag{46}$$

It is difficult to derive analytically an expression that gives the inclination of such orbits; however a numerical solution is developed.

5. Numerical solution algorithm

It is possible to introduce a numerical algorithm to find the intersection point of two satellites and the condition of the two orbits such that they constitute a Two-Way Orbit. This algorithm involves two consecutive steps: determining the intersection point and determining the orbit inclination.

5.1. Determine the intersection point

(38) and (43) can be solved numerically to find φ_1 and φ_2 as follows:

1. Assume a value for φ_1.
2. Get the corresponding ψ_1
3. Given the phasing parameters, F_n and F_d, evaluate ψ_2 using (43).
4. Evaluate the corresponding φ_2.
5. Check if φ_1 and φ_2 satisfy (38). If not, then repeat from step no. 1.

This algorithm will result in the values of φ_1 and φ_2 at the intersection point given the orbital shape, a, e, and ω.

5.2. Determine the orbit inclination

In this step, we use the Two-Way Orbit condition, (46), to find the orbit inclination. Given Ω_1, Ω_2 can be calculated as follows:

$$\Omega_2 = \Omega_1 - 2\pi\frac{Fn}{Fd}. \tag{47}$$

We then loop on all possible values of the inclination, and each time we check if the derived condition in (46)is satisfied or not. This step will result in all possible values for the inclination, i, completing the five orbital elements. There are many parameters that vary in the algorithm. One case is plotted in Fig. 4 and 5 where $F_n = 1$ and $F_d = 4$. The condition $\chi = 0$

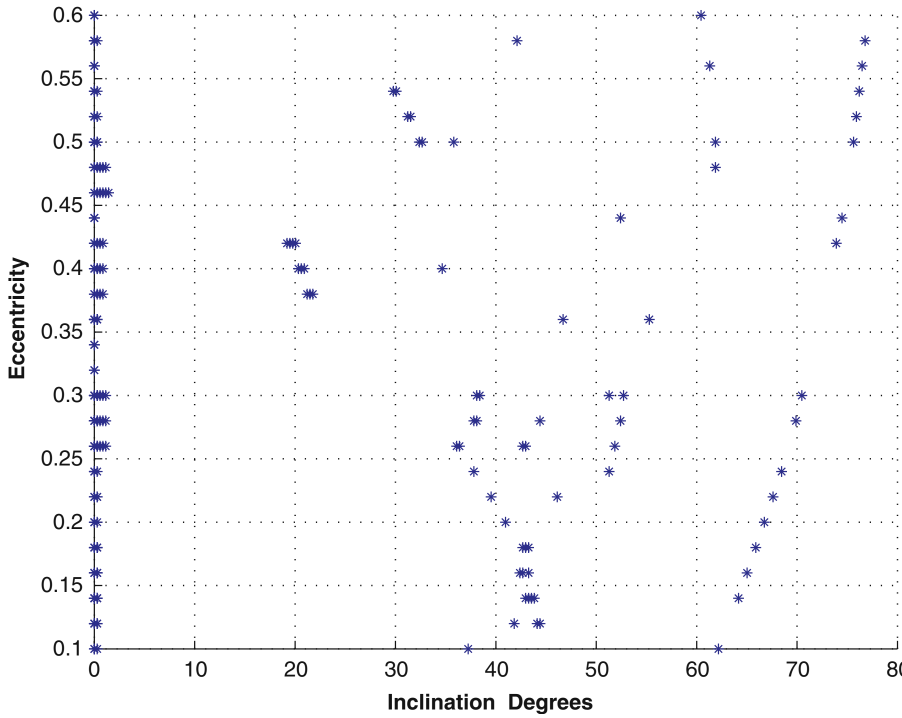

Fig. 4 General Two-Way Orbit: eccentricity vs. inclinations for different values of a

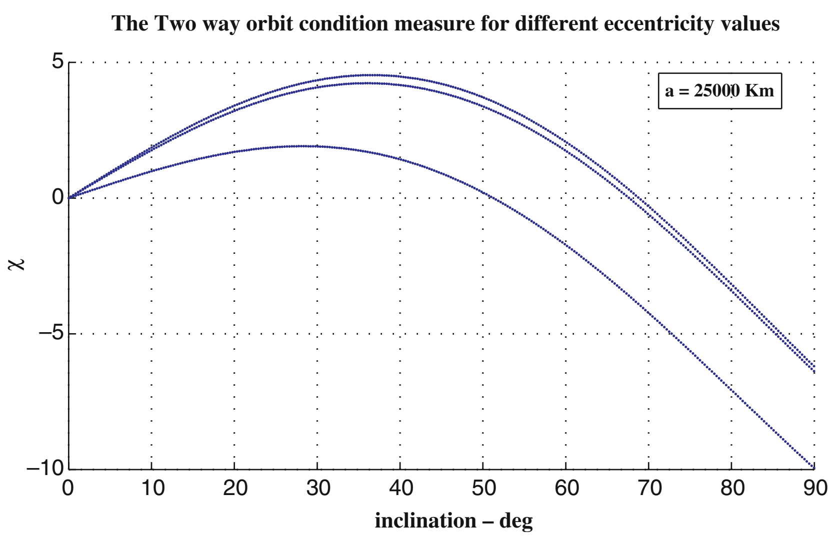

Fig. 5 General Two-Way Orbit: The χ values vs. inclination for different values of e

satisfaction is investigated and the variation of χ with different values for the eccentricity is plotted in Fig. 5.

For a Flower Constellation we have two phasing conditions. The first is used in calculating the intersection point as discussed above. The second is used to calculate Ω_2, (47) [So all the calculated satellites constitute a Flower Constellation.].

6. Conclusions

In this paper, the concept of Two-Way Orbits is investigated. The special case of two compatible orbits with a relative trajectory that intersects itself at the perigee of one orbit and the apogee of the other orbit is solved analytically. The general case of the two compatible orbits with a relative trajectory intersecting at two general points on their orbits with tangent ground tracks is formulated and solved numerically. The case of two satellites in a Flower Constellation is investigated and results demonstrated the possible existence of general Two-Way Orbits in Flower Constellation set.

References

Mortari, D., Wilkins, M.P., Bruccoleri, C.: The Flower Constellations. J. Astronautical Sci. Special Issue: The John L. Junkins Astrodynamics Symposium, No. 1 & 2, Vol. 52, pp. 107–127, January–June (2004a)

Sidi, M.J.: Spacecraft dynamics and control, a practical engineering approach. Cambridge University Press, NY (1997)

Mortari, D., Abdelkhalik, O., Bruccoleri, C.: Synodic and Relative Flower Constellations with Applications for Planetary Explorations. Paper AAS 05-151 of the 2005 Space Flight Mechanics Meeting Conference. Copper Mountain, CO, January 23–27 (2005)

Mortari, D., Wilkins, M.P., Bruccoleri, C.: On Sun-Synchronous Orbits and Associated Constellations. Paper of the 6th Dynamics and Control of Systems and Structures in Space Conference, Riomaggiore, Italy, July 18–22 (2004b)

Park, K.J., Wilkins, M.P., Bruccoleri, C., Mortari, D.: Uniformly Distributed Flower Constellation Design Study for Global Positioning System. Paper AAS 04-297 of the 2004 Space Flight Mechanics Meeting Conference, Maui, Hawaii, February 9–13 (2004)

Park, K.J., Ruggieri, M., Mortari, D.: Comparisons Between GalileoSat and Global Navigation Flower Constellations. Paper of the 2005 IEEE Aerospace Conference, Big Sky, Montana, March 5–12 (2005)

Wilkins, M.P.: The Flower Constellations - Theory, Design Process, and Applications. PhD Thesis, Texas A&M University, October (2004a)

Wilkins, M.P., Bruccoleri, C., Mortari, D.: Constellation Design Using Flower Constellations. Paper AAS 04-208 of the 2004 Space Flight Mechanics Meeting Conference, Maui, Hawaii, February 9–13 (2004b)

Wilkins, M.P., Mortari D.: Constellation Design via Projection of an Arbitrary Shape onto a Flower Constellation Surface. Paper of the 2004 AIAA/AAS Astrodynamics Specialist Conference, Providence, Rhode Island, August 16–19 (2004c)

Carter, D.: When is the Groundtrack Drift Rate Zero. CSDL Memorandum ESD-91-020 (1991)